Kazan Federal University Interregional Olympiads

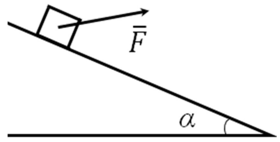
Physics

final stage (solutions/answers)

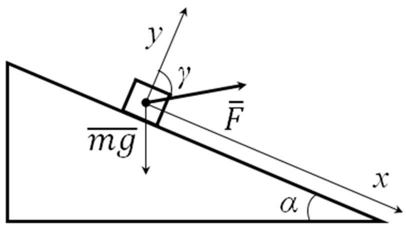
academic year 2022/23

11th (final) school year

Problem 1. (19 points) A bar of mass *m* lies on an inclined surface forming an angle α with the horizon. It is possible to apply the force *F* at the optimal angle in the plane of the figure. What is the minimal value of the force *F* that is necessary to move the body? The friction coefficient between the bar and the surface is μ. The external force is applied in a way that the bar moves translationally.



Possible solution:



Let's project the forces on the x- and y-axes and write Newton's second law of motion. To find the minimum force, equate the acceleration to zero.

$$\begin{cases} N + F\cos\gamma = mg\cos\alpha \\ 0 = mg\sin\alpha + F\sin\gamma - \mu N \end{cases}$$

$$\mu(mg\cos\alpha - F\cos\gamma) = mg\sin\alpha + F\sin\gamma$$

$$F = \frac{mg(\mu\cos\alpha - \sin\alpha)}{\sin\gamma + \mu\cos\gamma}$$

The numerator of this expression is constant. To reach the minimum it is enough to find the maximum value of the denominator

$$\sin\gamma + \mu\cos\gamma = \sqrt{1+\mu^2}\left(\frac{\mu\cos\gamma}{\sqrt{1+\mu^2}} + \frac{\sin\gamma}{\sqrt{1+\mu^2}}\right) = \sqrt{1+\mu^2}\cos(\gamma - \varphi)$$

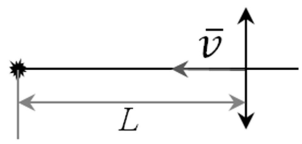
Thus, the denominator takes the maximum value $\sqrt{1+\mu^2}$. The minimum force required to move the bar from its place is

$$F_{min} = \frac{mg(\mu\cos\alpha - \sin\alpha)}{\sqrt{1+\mu^2}}$$

Evaluation Criteria:

Newton's second law is correctly written for all axes. 3 points per axis	6
An expression for the force is obtained from the equations	5
The minimum force is found (including using the derivative).	4
Correct expression for the minimum force is obtained	4

Problem 2. (18 points) A thin converging (collecting) lens moves with velocity *v* along its optical axis towards a static light source. The light source is located on the optical axis of the lens. The focal length of the lens is *F*. The distance from the light source to the lens at the moment of interest is *L* > *F*. What is the instantaneous velocity of the image in the laboratory frame of reference? Note: $(1+x)^{\gamma} \approx 1+\gamma x$ at $x \ll 1$.



Possible solution:

Let us use the reference frame associated with the lens. Denote by *d* the distance between the source and the lens, by *f* the distance between the lens and the image. According to the thin lens equation

$$\frac{1}{F}=\frac{1}{d}+\frac{1}{f}$$

$$f=\frac{Fd}{d-F}$$

The image velocity in this reference frame is $u = -\frac{df}{dt}$

There are two variants of finding the image velocity in a moving frame of reference.

1) Let's take the desired derivative directly and set $t = 0$

$$f=\frac{F(L-vt)}{L-vt-F}$$

$$-\frac{df}{dt}=-\left(\frac{-vF}{L-F}+\frac{FLv}{(L-F)^2}\right)=-\frac{F^2}{(L-F)^2}v;$$

2) Consider the thin lens equation in the first order by small time interval *dt*

$$f_0=\frac{FL}{L-F}$$

$$\frac{1}{F}=\frac{1}{L-vdt}+\frac{1}{f_0-udt}\approx\frac{L+f_0-(u+v)dt}{Lf_0-(Lu+vf_0)dt}=\frac{L+f_0-(u+v)dt}{Lf_0\left(1-\frac{(Lu+vf_0)}{Lf_0}dt\right)}\approx$$

$$\approx\frac{(L+f_0-(u+v)dt)\left(1+\frac{(Lu+vf_0)}{Lf_0}dt\right)}{Lf_0}\approx$$

$$\approx\frac{\left(\frac{(L+f_0)(Lu+vf_0)}{Lf_0}-(u+v)\right)dt}{Lf_0}+\frac{L+f_0}{Lf_0};$$

$$\frac{(L+f_0)(Lu+vf_0)}{Lf_0}-(u+v)=0$$

$$L^2u+vf_0^2=0$$

$$u=-\frac{vf_0^2}{L^2}=-\frac{F^2}{(L-F)^2}v$$

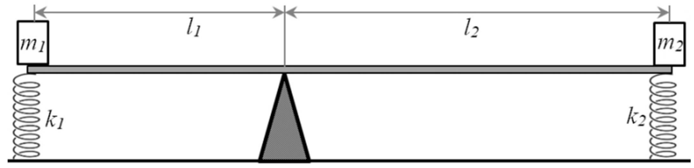
In the laboratory reference frame, the desired velocity will be

$$u'=\left(1-\frac{F^2}{(L-F)^2}\right)v$$

Evaluation Criteria:

Thin lens equation	3
The idea of the transition to the reference system associated with the lens. In the case of the alternative solution is scored	2
The correct image velocity is obtained by any method (in a laboratory or moving frame of reference) It is allowed to use the equation for linear magnification.	9
The correct answer is obtained	4

Problem 3. (20 points) At the ends of a weightless lever there are point masses m_1 and m_2 and attached weightless springs with stiffness k_1 and k_2. The distances from the ends of the lever to the fulcrum are l_1 and l_2, respectively. The lengths of the springs in an undeformed state are chosen so that the lever is in equilibrium in the horizontal position. Find the frequency of small vibrations of the lever after its small deviation from the horizontal position. The lever does not move away from its fulcrum in the course of the oscillations. The lengths of the springs are much longer compared with the amplitude of the oscillations.



Possible solution:

Consider first the equilibrium condition of the lever. Write the balance of moments with respect to the fulcrum

$$l_2 m_2 g + k_2 l_2 \Delta x_2 - l_1 m_1 g - k_1 l_1 \Delta x_1 = 0$$

Write the expression for the total mechanical energy of the system. The position of the lever and the velocity of all its points are uniquely determined by the inclination of the lever to the horizontal φ and the corresponding angular velocity $\dot{\varphi}$

$$E \approx \frac{m_1 l_1^2 \dot{\varphi}^2}{2} + \frac{m_2 l_2^2 \dot{\varphi}^2}{2} - m_1 g l_1 \sin\varphi + m_2 g l_2 \sin\varphi + \frac{k_1(\Delta x_1 - l_1 \sin\varphi)^2}{2} + \frac{k_2(\Delta x_2 + l_2 \sin\varphi)^2}{2}$$

In this equality we have taken into account that the length of the springs is much greater compared with the amplitude of the oscillations. For a small angle $\sin\varphi \approx \varphi$, expressed in radians. Then using a series expansion

$$E \approx \frac{m_1 l_1^2 \dot{\varphi}^2}{2} + \frac{m_2 l_2^2 \dot{\varphi}^2}{2} - m_1 g l_1 \varphi + m_2 g l_2 \varphi + \frac{k_1(\Delta x_1 - l_1\varphi)^2}{2} + \frac{k_2(\Delta x_2 + l_2\varphi)^2}{2} =$$
$$= \frac{(m_1 l_1^2 + m_2 l_2^2)\dot{\varphi}^2}{2} + (-m_1 g l_1 + m_2 g l_2 - k_1 \Delta x_1 l_1 + k_2 \Delta x_2 l_2)\varphi + \frac{k_1 {\Delta x_1}^2 + k_2 {\Delta x_2}^2}{2} + \frac{\varphi^2}{2}(k_1 l_1^2 + k_2 l_2^2)$$

The second term is zero due to the equilibrium condition.

Compare this expression with the energy of a one-dimensional harmonic oscillator:

$$E_h = \frac{m^* \dot{q}^2}{2} + \omega^2 \frac{m^* q^2}{2} + const$$

where q and $\dot{q}$ are generalized coordinate and generalized velocity, in this problem the angle and angular velocity. Then

$$E = \frac{(m_1 l_1^2 + m_2 l_2^2)\dot{\varphi}^2}{2} + \frac{(k_1 l_1^2 + k_2 l_2^2)}{(m_1 l_1^2 + m_2 l_2^2)} \frac{(m_1 l_1^2 + m_2 l_2^2)\varphi^2}{2} + const$$

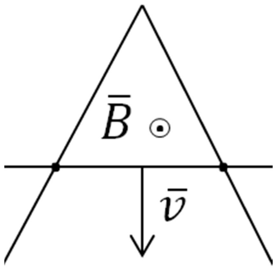
Thus the angular frequency and period of small oscillations are:

$$\omega = \sqrt{\frac{k_1 l_1^2 + k_2 l_2^2}{m_1 l_1^2 + m_2 l_2^2}}; \qquad T = 2\pi \sqrt{\frac{m_1 l_1^2 + m_2 l_2^2}{k_1 l_1^2 + k_2 l_2^2}}$$

Evaluation Criteria:

The equilibrium condition of the lever is correctly written	4
The total mechanical energy or restoring force for a system out of equilibrium position is correctly written	6
A small parameter expansion is performed	4
The energy or equation of vibration is reduced to the canonical form	4
The frequency (period) of oscillations is found	2

Problem 4. (21 points) A conductive jumper slides on a V-shaped conductive contour made of homogeneous wire. The jumper moves in a way that the V-shaped conductor together with jumper forms an isosceles triangle circuit, the side lengths of which increase with time. At the initial moment the area of the triangle is 0. The conductor and the jumper have the same resistance per unit length. The system is located in a constant and homogeneous magnetic field *B* perpendicular to the plane of the conducting circuit and jumper. The contact resistance of the jumper and the conductor is negligible. The inductance of the circuit is neglected.



a) (**15 points**) Find the dependence of jumper velocity *versus* time when the current in the circuit remain constant. (It is recommended to start with this question)

b) (+**6 points**) Find the jumper velocity *versus* time when the thermal power emitted by the circuit is constant.

In both cases, it is sufficient to present at least one type of dependence of velocity *versus* time.

Possible solution:

The area of a triangle *S* with fixed angles is proportional to the square of any side (Heron's equation) or height. The perimeter will obviously be proportional to any of the sides or heights. If the coordinate origin is placed at the vertex of V-shaped contour, the coordinate of the jumper center will coincide with the height of triangle:

$$S = ah^2, P = bh; \;\; a, b = const$$

According to Faraday's law, the electromotive force (EMF) induction in the circuit is

$$\xi = -\frac{d\Phi}{dt} = -\frac{BdS}{dt} = -2aBh\frac{dh}{dt} = -2aBhv,$$

where *v* is the jumper velocity.

The loop resistance is proportional to the perimeter $R = \gamma h$. The current modulus according to Ohm's law is

$$I = \frac{\xi}{R} = \frac{2aBhv}{\gamma h} = const \cdot v$$

Let's answer the question of problem a)

A constant velocity will correspond to a constant current

$$v(t) = const$$

Let's answer the question of problem b)

According to Joule-Lenz law, the heat power

$$P = \xi I = const \cdot hv^2$$

It is convenient to look for $h(t)$ dependence in the form of a power function

$$h = kt^{\beta}, \quad k = const$$
$$v = k\beta t^{\beta-1}$$

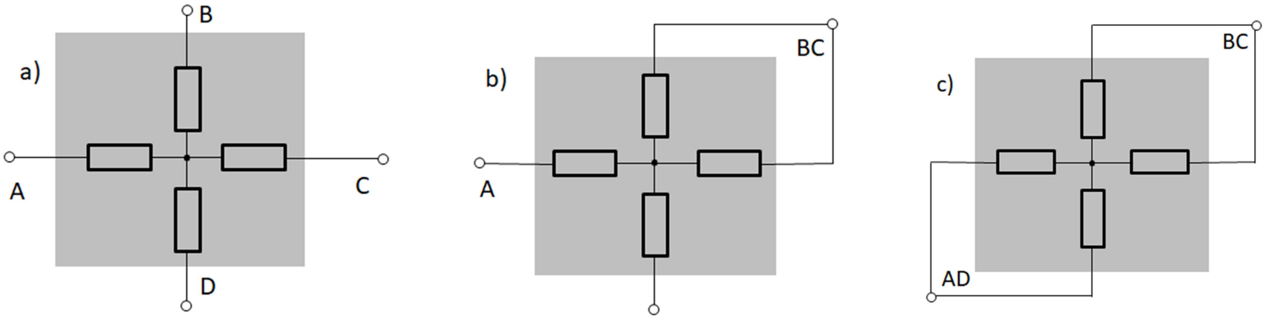
Then the constant power condition is

$$const = t^{\beta+2\beta-2}$$
$$\beta + 2\beta - 2 = 0; \qquad \beta = \frac{2}{3}$$
$$v(t) = const \cdot t^{-1/3}$$

Evaluation Criteria:

Faraday's law of electromagnetic induction.	2
Relation for resistance and perimeter of a circuit	2
Relation between the perimeter/height of a triangle and the area	4
Expression for current as a function of velocity	6
Correct answer to question a)	1
Expression for power *versus* height/perimeter and velocity	4
Correct answer to question b)	2

Problem 5. (22 points) Four identical resistors are connected as shown in the figure (see Fig. a), and soldered into a dielectric cube with high thermal conductivity. The resulting quadrupole is connected with connecting wires, whose resistance is negligibly small compared to that of the resistor, in all cases to the same ideal voltage source. When connected to terminals A and B, a current $I_1 = 1.00$ A flows through the source (see Fig. a). When connected to terminals A and BC, current $I_2 = 1.25$ A (see Fig. b). What current will flow through the source when it is connected to terminals AD and BC (see Fig. c)? The resistance of the resistors depends on the temperature according to a linear law. Consider that because of the high intensity of heat exchange inside the dielectric cube compared to the cube's heat exchange with the environment, the resistors' temperatures are almost equal at any connection option. The temperature and other environmental parameters are the same in all cases. The radiative heat transfer is neglected. All currents in the problem are assumed steady-state (after a long time after connection).



Possible solution:

In steady-state mode, the thermal power of the current is equal to the power of heat loss through the surface of the dielectric cube. Denote by U the voltage at the source, k the coefficient relating the temperature difference between the environment and the cube and the heat loss power through its surface.

$$\begin{cases} k\Delta T_1 = UI_1 \\ k\Delta T_2 = UI_2 \end{cases}$$

On the other hand, Ohm's law, taking into account the temperature dependence of resistance for the cases a) and b), is

$$\begin{cases} I_1 = \dfrac{U}{2R_0(1+\alpha\Delta T_1)} \\ I_2 = \dfrac{U}{1.5R_0(1+\alpha\Delta T_2)} \end{cases}$$

$$\begin{cases} I_1 = \dfrac{U}{2R_0\left(1+\dfrac{\alpha U I_1}{k}\right)} \\ I_2 = \dfrac{U}{1.5R_0\left(1+\dfrac{\alpha U I_2}{k}\right)} \end{cases}$$

Here U is the voltage at the source, R_0 resistance of one resistor at ambient temperature, α characterizes the dependence of resistance *versus* temperature.

Let's introduce the parameters $I_0 = \frac{U}{R_0}$ и $b = \frac{\alpha U}{k}$

$$\begin{cases} I_1 = \dfrac{I_0}{2(1+bI_1)} \\ I_2 = \dfrac{I_0}{1.5(1+bI_2)} \end{cases}$$

$b = \frac{4I_1 - 3I_2}{3I_2^2 - 4I_1^2} = \frac{4}{11} \approx 0.364$; $I_0 = \frac{6I_2 I_1 (I_2 - I_1)}{3I_2^2 - 4I_1^2} = \frac{30}{11} \approx 2.72$;

The current when connected to terminals AD and BC I_3 can be found from a similar equation.

$$\begin{cases} I_3 = \dfrac{U}{R_0(1+\alpha\Delta T_3)} \\ k\Delta T_3 = UI_3 \end{cases}$$

$$I_3 = \frac{I_0}{(1 + bI_3)}$$

Solving the quadratic equation, we obtain I_3

$$I_3 = \frac{-1 + \sqrt{4bI_0 + 1}}{2b} \approx 1.69\ A$$

Evaluation Criteria:

Equality of thermal power of current and power of heat loss through the surface.	4
Ohm's law for cases a) and b) taking into account the temperature dependence.	6
Introduction of parameters suitable for further solution of the problem.	3
Determination of the parameters introduced above from the known relation for currents.	4
The equation for the required current.	3
The correct answer is obtained.	2